SEM-4 (MATHEMATICS GENERAL)
(ANUP RAY)

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Date

Th 7: Let $G' = \{aH / a \in G\}$, $G'' = \{Ha / a \in G\}$ and $f: G' \to G''$, then

(i) f is bisective.

(ii) The number of distined right cosets of H in G = the number of distinct left cosets of H in G.

Proof: Let $f: G' \to G''$ be defined by $f(aH) = Ha^{-1}$ $\forall a \in G$

we shall show that f is well defined, since

$aH = bH \Rightarrow b \in aH \Rightarrow a^{-1}b \in eH = H$

$\Rightarrow a^{-1}(b^{-1})^{-1} \in H \Rightarrow Ha^{-1} = Hb^{-1}$ [by Th 4(iv)]

$\Rightarrow f(aH) = f(bH)$

Again f is one-one (injective) since

$f(aH) = f(bH) \Rightarrow Ha^{-1} = Hb^{-1}$

$\Rightarrow a^{-1}(b^{-1})^{-1} \in H$ [by Th. 4 (iv)]

$\Rightarrow a^{-1}b \in H \Rightarrow aH = bH$ [by Th 4(iv)]

Now for each $Ha \in G''$, $\exists a^{-1}H \in G'$ such that

$f(a^{-1}H) = H(a^{-1})^{-1} = Ha$

so f is onto i.e surjective.

Therefor f is bijective.

(ii) followe from (i) immediately.

Ex: Determine whether or not the following cosets of the subgroup $H = 5Z$ in the group $(Z, +)$ are equal:

(i) $-1 + H$ and $5 + H$

(ii) $3 + H$ and $2 + H$.

(Home work)

Index of a subgroup

If H be a sub-group of $(G, \cdot)$, then the number of distinct right (left) cosets of H in G is called the index of H in G and is denoted by $[G:H]$

Theorem 1 Let H be a subgroup of $(G, \cdot)$ and $f: Ha \to Hb$, where Ha and Hb are any two right cosets of H in G; $a, b \in G$. Then f is bijective.

Proof: Let $f: Ha \to Hb$ be defined by $f(ha) = hb$ $\forall h \in H$.

For $h_1, h_2 \in H$, $f(h_1 a) = h_1 b$ and $f(h_2 a) = h_2 b$

then $f(h_1 a) = f(h_2 a) \Rightarrow h_1 b = h_2 b$

$\Rightarrow h_1 = h_2$ by cancellation law in G.

$\Rightarrow$ f is one-one.

Again for each $hb \in Hb$, $\exists$ an element ha in Ha such that $f(ha) = hb$ $\forall h \in H$.

So f is onto.

Hence f is bijective.

Note: The above theorem also holds for any two left cosets of H in G.

Definition: Let H and K be two subgroups of a group G. Then we define $HK = \{hk / h \in H, k \in K\}$, then HK will be a non-empty subset of G.

Th. 2 HK is a subgroup of G if and only if $HK = KH$.

(Home work)